Student Perspectives on Equation: Constructing the Mathematical Object

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It has been some time since it was identified that student perspectives on equations and their use of the equals sign have not mirrored those of mathematicians. This paper describes some of the understandings of the equals sign displayed by secondary school students and seeks to analyse them in terms of properties of the constituent parts of equations. We find that students display a number of incomplete or pseudo-conceptions, and are sometimes influenced by representational aspects of the properties. A start is made on constructing a framework for understanding of the mathematical equation object.

Symbolic forms are ubiquitous in mathematics and hence understanding them is of prime importance. It is clear that for any written statement in symbolic form it is the meaning that the symbols take on in the mind of the reader that is of crucial importance. It has been suggested (Laborde, 2002), that objects in mathematics may be viewed from several perspectives, including a surface or perceptual one and a mathematical one, whereby the mathematical properties are understood. In this paper we address the hypothesis that for a mathematical equation it is the use of letters as variables and the '=' symbol that hold many of the mathematical properties that are attributed to the whole equation and hence understanding the mathematical object of equation requires the formation of these individual properties.

Previous research has suggested that to use equations in a versatile, mathematical way a strong symbol sense ought to be developed (Fey, 1990; Arcavi, 1994). While symbol sense has not been defined directly, it should include the knowledge that the correctness of an algebraic transformation may be checked, and that modelling particular phenomena requires a particular type of function. Further, a view of letters as encapsulated objects (Tall, Thomas, Davis, Gray, & Simpson, 2000) appears to be an important attribute. However, those who have written about symbol sense, such as Arcavi (1994) have limited this to behaviours that demonstrate good use of literal symbols, and this needs to be extended to include other symbols such as the equals symbol.

There is no doubt that many students struggle to attach meaning to many of the symbols used in mathematics. Mason (1987) suggests that a semiotic problem concerning the relationship between the sign and the signified, or the symbol and the symbolised is at the root of this. Further, the process of attaching appropriate meaning to mathematical symbols may be subverted by teaching that is heavily weighted in favour of instrumental learning (Skemp, 1976). Such a learning environment encourages a process-oriented view of mathematics where the object of study is not cognitively engaged, and hence pseudo-conceptions (Vinner, 1997) are more likely to occur. Once these pseudo-conceptions are in place they can be very resistant to change and may act as cognitive obstacles when a student is encouraged to perceive a mathematical object, such as an equation, via its properties.

There is clear evidence that students exhibit problems interpreting both the meaning of symbolic literals (Küchemann, 1981) and going beyond the separator operator interpretation of the equals symbol (Baroody & Ginsburg, 1983). It may be that these need to be addressed arithmetically before the concept of equation is introduced in an algebraic environment (Herscovics & Kieran, 1980). The study by Denmark et al. (1976) reports that first grade students were able to develop some flexibility in accepting the use of the equals symbol in a variety of arithmetic sentence structures (e.g., 3=3, 3+2=4+1, 5=4+1), achieving this by means of balancing activities and corresponding written identities. However the students still viewed the equals symbol primarily as an operator rather than a relational symbol. Herscovics and Kieran (1980) investigated student acceptance of an equivalent amount written in a variety of ways (e.g., 4+7=12–1) and found that they were able to accept and work quite comfortably with arithmetic identities containing multiple operations on both sides. It appears that the meaning of the equals symbol evolves as the mathematics encountered becomes more complex. In particular it changes from the intuitive ideas of sameness or counting the total found in arithmetic (Gelman & Gallistel, 1978) and the idea of the result of or answer to a procedure (Kieran, 1981), to a notion of the equivalence of algebraic statements with reflexive, symmetric and transitive properties. However, this process of change does not appear to come easily or quickly to many students.

When we ask the question 'What is an equation?' we may get a number of differing responses. One view of an equation is as a structural representation of a mathematical relationship between entities that are the objects of an algebraic and/or arithmetic system. One of the requirements for generating and adequately interpreting an equation structurally is a conception of the reflexive, symmetric, and transitive character of the equals sign as an equivalence relation. It is through these properties that the equals symbol conveys the concept of equivalence. On this topic of equivalence, Gattegno (1974, p. 83), stated:

> We can see that identity is a very restrictive kind of relationship concerned with actual sameness, that equality points at an attribute which does not change, and that equivalence is concerned with a wider relationship where one agrees that for certain purposes it is possible to replace one item by another. Equivalence being the most comprehensive relationship it will also be the most flexible, and therefore the most useful.

In spite of these ideas it seems that it is not that easy to specify precisely what an equation is or may be. Collins dictionary (Borowski & Borwein, 1989, p. 194) deals with the definition in this way:

> Equation, *n*. a formula that asserts that two expressions have the same value; it is either an *identical equation* (usually called an IDENTITY), which is true for any values of the variables, or a *conditional equation*, which is only true for certain values of the variables (the ROOTS of the equation).

Thus we have here two possible types of equation: a *conditional equation;* and an *identical equation* and, for example, $2x+1=6$ would be a *conditional equation,* but $2(2x+1)=4x+2$ would be an *identical equation*.

What are the distinctive properties of each of these responses to what an equation is that gives rise to them as a mathematical object? The purpose of this present research study is to investigate student perceptions of equation and to seek to answer these questions in terms of a framework for understanding equation. This paper presents data on how students understand the components of an equation, particularly the variables and the equals sign, and begins the process of constructing a framework attempting to map out student perspectives on equations.

Method

The results discussed here are from a pilot study that forms part of a larger cross-sectional study aimed at mapping out student use and understanding of the equals sign. Data in the pilot study were gathered during 2002 from 44 female and 37 male students who had studied mathematics at the year 12 level (range 15-18 years old) in that year. Two schools, both large multicultural, coeducational, public schools situated in the suburbs of Auckland, were chosen for the study. School A, which has a decile rating of 5, is more than twice as large as school B, which has a decile rating of 6 (the decile rating is a measure from 1-10 of socio-economic status of parents). Some of the students in school A had three years' of high school mathematics and some four years' experience, while all the students in school B had 4 years of high school mathematics. The top band of 10% of students in school A were not present as they had chosen to sit private examinations and had leave to study while the data was collected, but otherwise the classes used were considered to be mixed ability by their respective schools. A questionnaire containing 16 questions was given to the students to complete during one of their normal mathematics periods and the first named researcher supervised administration of the questionnaire. Only the results from two of the questions are considered here (see below for the questions).

Four students from school A were interviewed during the first week of school in 2003 for between 30 and 40 minutes, and these interviews were audiotaped. Firstly the students were asked about what made them feel confident about providing correct answers and how they checked whether they were correct. The students were then asked about their responses to questions 1 and 4, the questions analysed here. They were provided with a transcript of their responses to these questions from the questionnaire so that they could familiarise themselves with what they had written and reflect upon it. The interview protocol was semi-structured in that the questions were specifically constructed and a number of potential directions were anticipated depending on how the students responded. While the students were told that they were able to end the interview any time they desired, none of them asked to end it prematurely.

Results

It appears that there are two major conceptual components of an algebraic equation, namely the representation of variables as letters and the equals sign, and each of these can be perceived in a number of qualitatively different ways, leading to a range of possible perspectives on equations. In order to examine the way that the students viewed these main components they were given the following two questions:

- Q1 What does $y = 2x + 3$ mean to you?, and
- Q4 What does $2p + 3 = q$ mean to you?

When responding to these questions their perspective on what the letters or symbolic literals stand for was seen to be influencing their replies. For example, 50% of the students referred to the idea of 'having the same value as' rather than, say, being the same as. In Küchemann's (1980) analysis there is a clear difference in thinking between those who give letters a value from the start and those who are able to think of letters as having a specific but as yet unknown value. While it can be a little difficult to separate these two given limited data, there is a difference between the immediate evaluation of letters seen in comments such as:

> SC If either '*x*' or '*y*' are substituted for a numerical value, then the numerical value of the other can be worked out. (Q1)

PW Means when y = a specific number, and x = a specific number, the number on each side of "=" is the same. (Q1)

and the use of letter as specific unknown, a 'certain' or particular number which may be identified in responses such as:

RB That when p = a certain number q = a certain number. (Q4)
JF When y equals a certain number then 2 times another number +3 will equal that. (Q1)
RT When 'p' is double and 3 is added to this it is the same number as 'q' (Q4)
JL That two times a number plus 3 is the same as another number (Q1)
AR An unknown number represented by 'y' is the same as two of another unknown 'x' + 3. (Q1)
EAM It means, the number y represents is the same as the number that the number x represents times two plus three. (Q1)
RB That there are 2 unknown numbers and that this equation is how they're related (Q1)

However, some responses appeared to be at a higher level according to Küchemann's classification, and demonstrated an understanding of letter as generalised number, or possibly even variable.

JC That y is equal to what x is times by 2 and added to 3 (Q1)
RB y is equal to $2x + 3$. When y changes x changes and when x changes y changes. (Q1)
BL It means that for whatever x value you put into the equation y will be equal to the answer when solved. (Q1)
BL When p is substituted for a value and then multiplied by 2 and added to 3 it will be equal. (Q4)
N that 'y' has the same numerical value as '$2x + 3$' (Q1)
RBa It means that the term '$2x + 3$' has the same value as 'y'. (Q1)
RBa Means that twice 'p' with '3' added to it have the same values as 'q'. (Q4)

Some of these comments give the idea that when the letters are used in calculations, or when they 'change' that the same result or 'answer' is produced. The use of words such as 'the same value(s)' was quite common, and while we can not always be sure that the students are in fact thinking of more than a single value for each letter, sometimes we are assisted by their further comments. For example, BL refers to 'whatever x value you put into the equation' and RBa uses the phrase "the same values as q' in the plural in Q4, showing that she is thinking that more than one value is possible. This perspective helps students to move from focussing on the values of the variables to a relationship between the variables themselves, as seen in RB's comment "When y changes x changes and when x changes y changes."

Just as the letters in an equation may be seen in a number of different lights, so too may the equals sign. The first view of this sign which is usually formed, based on arithmetic experiences, is that given by Kieran (1981) where the '=' sign is seen as signifying the result or answer to some calculation. To others it may signify the idea of a conditional equality (i.e. sometimes equal), while some may see it as expressing an identical equality (always equal), and finally for a few, it may be seen as signifying an equivalence relation. While these perspectives are not mutually exclusive, it seems students are more likely to appreciate the first and second than the third and fourth. The idea of giving a result or answer may be inferred from the comments of RT we carry out a procedure and "this is the same number as q". RB similarly even reverses the y=2x+3 to the process-oriented format (Thomas, 1994) $2x+3=y$ so that y is the result of the calculation. On the other hand, comments presented above by JF "number…will equal that", JL and AR "number…is the same as", PW "numbers on each side of the '=' is same", and BL "will be equal to the answer", seem to indicate a view of conditional equality. We note that this idea occurs in all the different letter perspectives above.

It is to be expected that the concept of an identical equality will emerge gradually and in line with this some of the students appear to have thinking that could be described as intermediate between conditional equality, with its equality of resulting values, and an identical equality, with the structure of equivalence of expressions.

> RR An equation indicating that after substituting x with a number, $2x + 3$ will be the same as y. (equal to). (Q1)
> BP y is equivalent to $2x + 3$, it is the same value (Q1)

Here RR has the idea of "will be the same as", rather than have the same value as, but this arises after "substituting x with a number". Similarly BP is states that "y is equivalent to $2x + 3$", but then says that it is because they have the same value. In contrast the fully fledged equivalence perspective may be seen in some of the students' comments:

> AR It means y is equivalent to $2x + 3$ or x is equivalent to $\frac{y - 3}{2}$ (Q1)
> RT That 'y' is the same as '$2x + 3$', they are equivalent (Q1)
> SW y can be used instead of $2x + 3$ (Q1)
> CE y is the same as $(2x + 3)$ (Q1)
> LS y and $2x + 3$ are the same. (Q1)

In these cases it is no longer just that the two expressions have the same value, but they now are seen as equivalent, so one can be "used instead of" the other, "is the same as" it, or is "equivalent to it". There is a much stronger structural tone to these comments related to the equals sign, and AR is even able to state that x is equivalent to $\frac{y - 3}{2}$.

When we were devising the questions for this study we wondered whether it was a good idea to use y and x as variables since there could likely be interference from graphical schemas where the use of these letters is so common, thus skewing our results. However, simply because the letters are so common we thought it good to obtain a perspective on these, along with another set, p and q. In the event, some students did refer to graphs in their explanations, but only 10% did so. Evidence that the letters x and y do evoke a particular representational context was provided by answers where the response to Q1 was graphical but that to Q4 was not. For example:

> SK (Q1) It is a graph representation or it means the value y is equal to the value $2x + 3$ like linear equation.
> SK (Q4) It means the value of q is equal to the value of $2p + 3$. And like question 1) it can also be a equation for graph representation (or like linear equation).
> LJ (Q1) This is a formula for a graph. To use y= to work out they intercept and the 2x to work out the gradient.
> LJ (Q4) Somehow you need to solve the values for p and q.
> CS (Q1) It is a straight line graph which cuts the y-axis at y=3.
> CS (Q4) An equation made up by a group of data. p and q represent numbers which are to be figured out. But they can never be figured out in this case as we need 2 out of 3 numbers.
> IY(Q1) A line on a y and x axis with a gradient of 2 and y intercept of (0, 3).
> IY (Q4) $2p$ plus 3 equals q.

However, this disparity of view was not always the case, and the influence of a graphical perspective was strong enough with some students to carry over to the second equation too. This seems to be evidence that the representational context in which linear equations are met most often may be strong enough to overcome other influences, such as the letters involved. Certainly this seems to be true for the graphical representation.

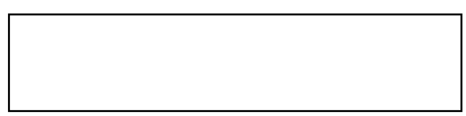
> SMc (Q1) It means any y-coordinate on a graph can be obtained by plugging in the x number
> SMc (Q4) Straight line graph. q is the same as y would be. Same as q=2p+3

BH (Q1) A straight line with gradient 2 y -intercept of 3, x-intercept of $^{-2}\!/_{3}$. As x increases y does so by double plus 2
BH (Q4) same as Qn 1)
SL (Q1) An equation of a line
SL (Q4) Line equation

We considered whether we could discern any evidence in the student comments of a move towards a view of the equals sign as an equivalence relation. Of course, demonstrating an appreciation of the symmetric, reflexive, or transitive properties of itself is not the same as constructing the equivalence relation. We did not expect to observe the reflexive property due to its subtlety, or the transitive property since there are only two expressions present in the questions, but we did find some evidence of the symmetric property. For example, SMc's remark above that $2p+3 = q$ is the "Same as $q=2p+3$", RY's writing "$q = 2\times p$ (unknown) + 3", and EA-M's "The number q is the number p timesed by two 3" where the original equation is reversed appear to constitute examples of the use of the symmetric property.

Discussion

Laborde (1995, 2002) has discussed with reference to geometry the nature of the difference between a drawing and a figure. She explains that the former is physical and perceptual, while the latter is theoretical and mathematical. Thus when we perceive an object we may gain a surface view, but in order to get a mathematical perspective of what it represents we have to interpret what we see. (c.f., Booth & Thomas, 2000). This interpretation of the symbolisation or representation takes place via identification of the object's properties, which are often the underlying invariants. One way that mathematical objects arise is by reflective abstraction, and the synthesis abstracted properties into a new object, the mathematical one. For example, when first learning geometry, we may be given a figure such as this:



We may be told that it is a rectangle, but this is simply a naming exercise and our conception of a rectangle will be based on properties obtained only by perception, or by the action of surface or deep observation (Thomas & Hong, 2001). Even when we have seen lots of objects which we begin to recognise as belonging to the same class of rectangles, we have not constructed the mathematical object of rectangle. It is only when we have discovered the properties that constitute the mathematical object of rectangle (enabling us to decide what makes another object 'not a rectangle'), namely that it has two pairs of opposite sides equal and four 90˚ angles, that we have constructed the mathematical conception. A pseudo-conception of rectangle (Vinner, 1997) can lead to errors such as the common one when students are shown a square and asked if it is a rectangle, and reply no, because their perception is that it is not in the class. They have not reasoned that it satisfies the required properties. The ability to see squares as a subset of rectangles is mediated by the understanding of the relationship between their properties.

Let us consider the corresponding situation with regard to equation. We can form a surface recognition of an equation based on the surface observation that it contains an '=' sign. However, when faced with questions about such objects with this feature students often reveal a pseudo-conception, exhibiting a lack of depth to their understanding. We can probe understanding by asking questions such as whether the following are equations:

$$x = x \qquad 0 = 1 \qquad \frac{2x-6}{x-3} = \pm 1 \qquad y{=}f(x), \qquad \text{etc.}$$

Unlike the rectangle though, we may discover that an equation is not so easy to tie down in terms of the properties that define the mathematical object. What the research presented here suggests is that the mathematical equation object is a gestalt object, with the parts comprising the arithmetic numbers, the variables, the operators and the equals sign. One's mathematical understanding of these constituent parts becomes welded into a coherent whole, the mathematical equation, greater than the sum of the parts..

We have concentrated here on the role of the variables and the equals sign in this process, and while there is certainly more to be said, we have distinguished a number of differing perspectives of each that contribute to a variety of perspectives on equation. These are summarised in the provisional outline framework for equation in Figure 1.

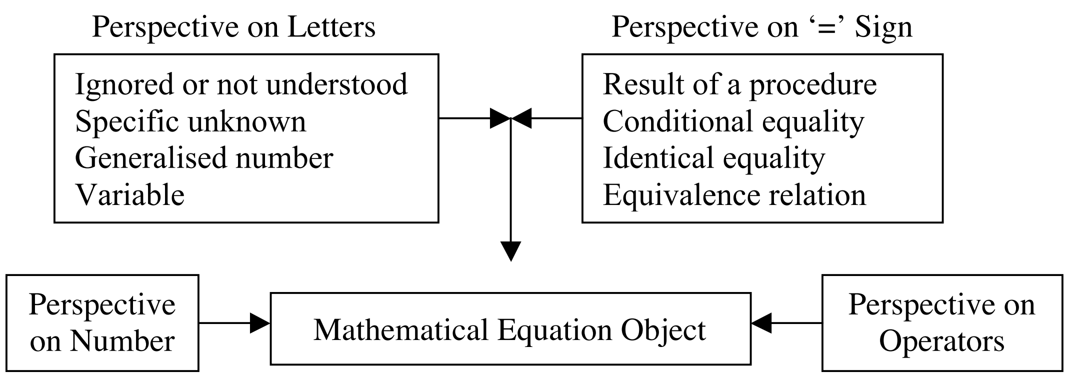


Figure 1. An outline framework of the mathematical equation object.

We note that one reason why students may lack a view of the equals sign as an equivalence relation is that teachers often use the symmetric, reflexive or transitive properties of equals without making these explicit. Consider the following examples of this. In solving an equation we may go from $x + 6 = 3x + 1$ to $2x + 1 = 6$ by using the symmetric property, or we may reason along the lines that $y = 2x + 1$ therefore when y=0, $2x + 1 = 0$, employing the transitive property to do so. However, we may not highlight these properties explicitly in either case. If students have a view of the equals sign as signifying the result of a procedure or as conditional equality then they may not have constructed these properties and will not be able to interact fully with the mathematical equation object. Certainly there is much deliberate effort required to assist students to enrich their perspective on equation.

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